Research Proposal

Strongly surviving critical contact processes, exit times limit laws revisited, and the ant in a labyrinth.

Achillefs Tzioufas

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Introduction

Percolation theory aims to understand phenomena raging from epidemics and fires in orchards, to the distribution of matter in the universe. Percolation on integer lattices, $\mathbb{Z}^d$, is the canonical model for describing flow in disordered porous media (Grimmett [9, 10] and Kesten [11]). It is the simplest model to exhibit a *phase transition*, i.e. increasing local connectivity rules results in passing from connected clusters of exponentially small size, to the emergence of a (unique) *infinite connected cluster*. It is for this reason that the percolation cluster constitutes the canonical paradigm of an infinite random graph. The *Contact Process* (Liggett [14, 15]) and its discrete time analogue, viz. *oriented percolation*, are the canonical percolation models incorporating a notion of either time, when modeling an epidemic or, when modeling flow in a random medium, gravity.

Random walks constitute the archetypical mathematical formalization of disordered motion resulting from successive random increments, and are a central object in probability theory since the beginnings of the subject with the investigations of De Moivre, Laplace, Bernoulli, Pascal, among others. The main references on the topic, whose everlasting influence cannot be overestimated, are those of Feller [Chpts. III and XIV, [8]] and Spitzer [18], whereas, from the plethora of more recent ones, we refer to Lawler [12], Lawler and Limic [13] due to being closer to our scope, and also to the amusing introduction of Doyle and Snell [7]. We will exploit connections of random walks with the so-called model of *Brownian motion* and *electrical networks*.

Pedesis is the phenomenon of chaotic displacements of small particles suspended in a liquid or in a gas resulting for collisions with the molecules of the medium, the existence of which was empirically confirmed by botanist Robert Brown in 1827, although known earlier references to it date back to scientific poem of poet and philosopher Lucretius "On the Nature of Things" (c. 60 BC). Its precise detailed explanation, offered by Albert Einstein in one of his Annus mirabilis papers, 1905, served as definitive confirmation that atoms and molecules actually exist. The model of *Brownian motion* (or the *Wiener process*) is one of the most important ones in the theory of random processes, despite that there exist several mathematical models for the phenomenon. Brownian motion arises as the scaling limit of the random walk in the limit of the lattice spacing going to zero.

Furthermore, we shall exploit the intimate, deep connection amongst random walks and the classical physics subject of *electrical networks*. This connection has had profound consequences and, indeed, the interplay between the two has been proven mutually beneficial. It allows for probabilists to draw on a large body of ways of thought and well established methods from the physics literature, most prominently involving considerations of energy as the Thomson and Dirichlet principles. Whereas it also provides with insights and interpretations to quantities and physical laws pertaining to electrical networks, as for example that of effective resistance and its monotonicity law (cf. Doyle and Snell [7]).

Part 1 of this proposal regards work on the Contact Process. Part 2 regards work on a the random walk in connection with Brownian motion. Part 3 lies in the interface of percolation and random walks in connection with electrical networks. In what follows we elaborate on each of the parts of the project.

Specific problems

Summary

An outline of the threefold proposed project is summarized as follows. Part 1 concerns strong survival of the critical contact process from infinite configurations on $\mathbb{Z}^d$. In Part 2 we show an elementary approach to certain limit laws for random walks. In Part 3 we intend to investigate the scaling limit of escape probabilities for the random walk on the infinite percolation cluster of $\mathbb{Z}^d$, $d \geq 2$.

1. Since the foundational work of Bezuidenhout and Grimmett [1] it is known that the contact process on integer lattices possesses no more than one non-trivial invariant measure. This is a consequence of that the process from the fully occupied configuration at criticality converges to the empty configuration in a weak sense. In this work we show that, nevertheless, each finite subset is at a fully occupied state for arbitrarily large times with probability one. This

is a manifestation of that weak convergence results fail to capture phenomena that almost sure ones do. The method of proof we use for extending this consequence of the non-negative asymptotic speed result in dimensions higher than one relies on techniques developed in Bezuidenhout and Grimmett [2] for establishing there that the critical exponent associated to the Lebesgue integral of the occupied region in the graphical representation of the process assumes value which is greater than or equal than that predicted by mean field theory (2, in the logarithmic sense).

We investigate upon extensions and consequences of the aforementioned result. As a first consequence we show that a certain interacting particle system, the idle contact process at criticality, exhibits strong survival with positive probability (w.p.p.). In this process, at the outset, an idle particle is placed on each site of $\mathbb{Z}^d$ other than the origin, which is inhabited by an excited particle instead. Idle particles are set to diffuse their descendancy according to a contact process at criticality, if ever excited. Once excited an idle particle immediately initiates reproduction, as well as attempts to excite its neighbors at a certain rate before dying. Progeny of different particles evolves otherwise independent of one another. The proof of that this process exhibits strong survival w.p.p. is a consequence of an extension of the result mentioned in a) in the case that the initial configuration is that of an infinite (supercritical) site percolation cluster and by means of offering a new basic coupling for independent contact processes. To show the former mentioned extension we rely on that an infinite percolation cluster does not fail to include an infinite subset of any fixed on the outset infinite collection of points, a consequence of its uniqueness by basic ergodicity properties. Two other corollaries regarding weakened initial conditions under which the result in a) remains valid for configurations that contain an infinite number of infinite, finite width strips of vacant sites, and for configurations distributed according to the invariant measure of a highly supercritical contact process are shown as byproducts.

2. This work is concerned in revisiting a problem of quite long mathematical history. The so-called *absorption problem* regards asymptotics of the random, almost surely finite, times the module of the walk attains new maxima values,

i.e. the times the walk exits symmetric intervals about its origin[1] (Theorem 2.13 Revesz [17]). As an alternative to the various available interesting analytical methods, we show a new approach that is elementary in its entirety. This approach relying on Laplace transformations apparatus which, leaning on the so-called continuity theorems, commonly attributed to P. Levy, yields neat proofs. We find this approach of interest in its own right due to its overall elementary and simple nature. An additional interesting feature of this approach is that it yields as byproduct a connection with first-passage times and the stable law of order $1/2$ limit for first-passages times is retrieved.

We exploit this result to show an elementary proof to that symmetric planar random walks exit times from spheres and partial maxima values in ℓ_1 and in ℓ_∞ distances, under appropriate in either case rescaling, possess associated asymptotic distributions, which are identified in explicit form and associated to functionals of standard planar Brownian motion. A link to the solution to associated boundary value problems of the simplest type is also pointed out.

3. We consider the Simple Random Walk on an infinite percolation cluster model, *viz.* the so-called *ant in a labyrinth*. Results concerning the scaling limit of this process have been of central interest in the probability literature for years (cf. with the recent survey of Biskup [3]). The problem under consideration, originally proposed in the monograph of Kesten [11], concerns the limiting behavior and asymptotics of the effective resistance of the electrical network associated with the cluster of supercritical bond percolation. By elementary considerations involving Kirchhoff's and Ohm's laws the effective resistance and voltages on the vertices of the network correspond to determining certain passage times for the random walk on this cluster and the construction of the harmonic function respectively. We investigate open problems 12 and 13 from Kesten [11]. The conjectures effectively concern extending to the case of a the walk on the random environment of an infinite percolation cluster the well-known asymptotic scaling result for escape probabilities of the Random Walk (see for instance Proposition 2.16 in [16], or Lemma 22.1 in [17]). To resolve these problems, we point out to known asymptotics regarding the geometry of the cluster, in particular, regarding

[1] the problem's nomenclature derives from the equivalent perspective of the walk restrained on intervals with absorbing endpoints

the boundary to volume ratio, see Theorem 8.99 in Grimmett [9]. The reason we believe this ratio hints the right scaling for the quenched escape probabilities correctly is that it seems to represent (intuitively) the quote of the escape paths remaining open (volume) to those blocked (boundary).

References

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